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Nonlinear branching processes with immigration

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Abstract. The nonlinear branching process with immigration is constructed as the pathwise unique solution of a stochastic integral equation driven by Poisson random measures. Some criteria for the regularity, recurrence, ergodicity and strong ergodicity of the process are then established.

Key words and phrases. Nonlinear branching process, immigration, stochastic integral equation, regularity, recurrence, ergodicity, strong ergodicity.

1 Introduction

Markov branching processes are models for the evolution of populations of particles. Those processes constitute one of the most important subclasses of continuous-time Markov chains. Standard references on those processes are Harris (1963) and Athreya and Ney (1972). The basic property of an ordinary linear branching process is that different particles act independently when giving birth or death. In most realistic situations, however, this property is unlikely to be appropriate. In particular, when the number of particles becomes large or the particles move with high speed, the particles may interact and, as a result, the birth and death rates can either increase or decrease. Those considerations have motivated the study of nonlinear branching processes. On the other hand, a branching process describes a population evolving randomly in an isolated environment. A useful and realistic modification of the model is the addition of new particles from outside sources. This consideration has provided the stimulation for the study of branching models with immigration and/or resurrection.

Let $\{r_i : i \geq 0\}$ be a sequence of nonnegative constants with $r_0 = 0$ and $\{b_i : i \geq 0\}$ a discrete probability distribution with $b_1 = 0$. A continuous-time Markov chain is called a nonlinear branching process if it has density matrix $R = (r_{ij})$ given by

$$r_{ij} = \begin{cases} r_i b_{j-i+1} & j \geq i+1, i \geq 1, \\ -r_i & j = i \geq 1, \\ r_i b_0 & j = i-1, i \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

A typical special case is where $r_i = \alpha i^\theta$ for $\alpha \geq 0$ and $\theta > 0$, which reduces to the ordinary linear branching process when $r_i = \alpha i$. Let $\gamma \geq 0$ and let $\{a_i : i \geq 0\}$ be another discrete probability distribution satisfying $a_0 = 0$. A continuous-time Markov chain is called a nonlinear branching process with resurrection if its density matrix is given by

$$\rho_{ij} = \begin{cases} r_i b_{j-i+1} & j \geq i+1, i \geq 1, \\ -r_i & j = i \geq 1, \\ r_i b_0 & j = i-1, i \geq 1, \\ \gamma a_j & j > i = 0, \\ -\gamma & j = i = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1.2)$$

Here the resurrection means that at each time when the process gets extinct, some immigrants come into the population at rate γ according to the distribution $\{a_i\}$. By a nonlinear branching process with immigration we mean a Markov chain with density matrix $Q = (q_{ij})$ given by

$$q_{ij} = \begin{cases} r_i b_{j-i+1} + \gamma a_{j-i} & j \geq i+1, i \geq 0, \\ -r_i - \gamma & j = i \geq 0, \\ r_i b_0 & j = i-1, i \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.3)$$

In this model, the immigrants come at rate γ according to the distribution $\{a_i\}$ independently of the inner population.

The purpose of this paper is to investigate the construction and basic properties of the nonlinear branching process with immigration defined by (1.3). Let

$$m = \sum_{j=0}^{\infty} j a_j, \quad M = \sum_{j=0}^{\infty} j b_j,$$

which represent the birth mean and immigration mean of the process, respectively. Moreover, we introduce the functions

$$F(s) = \sum_{i=0}^{\infty} a_i s^i, \quad A(s) = \gamma(1 - F(s)), \quad G(s) = \sum_{i=0}^{\infty} b_i s^i, \quad B(s) = G(s) - s, \quad s \in [0, 1].$$

Let q be the smaller root of the equation $G(s) = s$ in $[0, 1]$. We sometimes denote r_i by $r(i)$ for notational convenience.

Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a probability space satisfying the usual hypotheses. Let $\{p(t)\}$ and $\{q(t)\}$ be (\mathcal{F}_t) -Poisson point processes with characteristic measures $dum(dz)$ and $\gamma n(dz)$, respectively. We assume $\{p(t)\}$ and $\{q(t)\}$ are independent of each other. Let $N_p(ds, du, dz)$ and $N_q(ds, dz)$ be the Poisson random measures associated with $\{p(t)\}$ and $\{q(t)\}$, respectively. Given an \mathbb{N} -valued \mathcal{F}_0 -measurable random variable X_0 , let us consider the stochastic integral equation

$$X_t = X_0 + \int_0^t \int_0^{r(X_{s-})} \int_{\mathbb{N}} (z-1) N_p(ds, du, dz) + \int_0^t \int_{\mathbb{N}} z N_q(ds, dz). \quad (1.4)$$

Let $\zeta = \lim_{k \rightarrow \infty} \tau_k$, where $\tau_k = \inf\{t \geq 0 : X_t \geq k\}$. The above equation only makes sense for $0 \leq t < \zeta$. We call ζ the explosion time of $\{X_t\}$ and make the convention $X_t = \infty$ for $t \geq \zeta$. We say the solution is non-explosive if $\zeta = \infty$. As a special case of (1.4) we also consider the equation

$$X_t = X_0 + \int_0^t \int_0^{r(X_{s-})} \int_{\mathbb{N}} (z-1) N_p(ds, du, dz). \quad (1.5)$$

We now state the main results of the paper.

Theorem 1.1 *There exists a pathwise unique solution to (1.4). Moreover, if the solution to (1.5) is non-explosive, then so is the solution to (1.4).*

Theorem 1.2 *Let $\{X_t\}$ be the solution to (1.4) and let $Q_{ij}(t) = P(X_t = j | X_0 = i)$. Then $Q_{ij}(t)$ solves the Kolmogorov forward equation of Q .*

Theorem 1.3 *The solution to (1.4) is the minimal process of Q and the solution to (1.5) is the minimal process of R .*

Theorem 1.4 *The density matrix R is regular if and only if Q is regular.*

Theorem 1.5 (1) *If $M \leq 1$, then Q is regular.*

(2) *Suppose that $\sum_{i=1}^{\infty} r_i^{-1} < \infty$. Then Q is regular if and only if $M \leq 1$.*

(3) *Suppose that $1 < M \leq \infty$ and $r_i = \alpha i^\theta$ for $\alpha > 0$ and $\theta > 0$. Then Q is regular if and only if for some $\varepsilon \in (q, 1)$, we have*

$$\int_{\varepsilon}^1 \frac{1}{B(s)} \left(\ln \frac{1}{s} \right)^{\theta-1} ds = -\infty.$$

In the following three theorems, we assume $\gamma r_i b_0 > 0$ for every $i \geq 1$, so the matrix Q is irreducible.

Theorem 1.6 (1) *Suppose that $m < \infty$, $M < 1$ and $\lim_{i \rightarrow \infty} r_i = \infty$. Then the nonlinear branching process with immigration is recurrence.*

(2) *Suppose that r_i is increasing and there exist constants $\alpha > 0$ and $N > 0$ such that $r_i/i \geq \alpha$ holds for each $i > N$. Then the nonlinear branching process with immigration is recurrent if $M \leq 1$ and*

$$J := \int_0^1 \frac{1}{\alpha B(y)} \cdot \exp \left[- \int_0^y \frac{A(x)}{\alpha B(x)} dx \right] dy = \infty.$$

(3) *Suppose that $M > 1$. Then the nonlinear branching process with immigration is transient.*

(4) Suppose that r_i is increasing and there exist constants $\alpha > 0$ and $N > 0$ such that $r_i/i \leq \alpha$ holds for each $i > N$. Then the nonlinear branching process with immigration is transient if $M \leq 1$ and

$$J := \int_0^1 \frac{1}{\alpha B(y)} \cdot \exp \left[- \int_0^y \frac{A(x)}{\alpha B(x)} dx \right] dy < \infty.$$

Theorem 1.7 (1) If $m < \infty$, $M \leq 1$, r_i is increasing and $\sum_{i=1}^{\infty} r_i^{-1} < \infty$, then the nonlinear branching process with immigration is ergodic.

(2) Suppose that $r_i = \alpha i^\theta$ for $\alpha > 0$ and $\theta \geq 1$. Then the recurrent nonlinear branching process with immigration is ergodic if and only if

$$\int_0^1 \frac{A(s)}{\alpha B(s)} \left(\ln \frac{1}{s} \right)^{\theta-1} ds < \infty. \quad (1.6)$$

(3) If $m < \infty$, $M < 1$ and $\liminf_{i \rightarrow \infty} r_i/i > 0$, then the nonlinear branching process with immigration is exponentially ergodic.

Theorem 1.8 (1) If $m < \infty$, $M < 1$, r_i is increasing and $\sum_{i=1}^{\infty} r_i^{-1} < \infty$, then the process is strongly ergodic.

(2) Suppose that $r_i = \alpha i^\theta$ for $\alpha > 0$ and $\theta > 1$. Then the nonlinear branching process with immigration is strongly ergodic if and only if

$$\int_0^1 \frac{1}{\alpha B(s)} \left(\ln \frac{1}{s} \right)^{\theta-1} ds < \infty. \quad (1.7)$$

(3) If $\sum_{i=1}^{\infty} r_i^{-1} = \infty$, then the nonlinear branching process with immigration is not strongly ergodic.

The nonlinear branching process with resurrection defined above was introduced by Chen (1997), who studied the problems of uniqueness, recurrence and ergodicity of the process. The model has attracted the attention of a number of authors. In particular, Zhang (2001) gave criteria for strong ergodicity of the process. Chen et al. (2005) and Pakes (2007) established some criteria for their regularity and uniqueness. Chen (2002) studied some interesting differential-integral equations associated with a special class of nonlinear branching processes and gave some characterizations of their mean extinction times. Chen et al. (2006) established a Harris regularity criterion for such processes. The existence and uniqueness of linear branching processes with instantaneous resurrection were studied in Chen and Renshaw (1990). However, most of the study of models with immigration have been focused on linear branching structures. The branching process with immigration was studied in Karlin and Taylor (1975), who gave a characterization of the one-dimensional marginal distributions of the process starting from zero. An ergodicity criterion for the process was given in Yang (1975). Li and Chen (2006) established some recurrence criteria for linear branching processes with immigration and resurrection.

The first three theorems above give constructions of nonlinear branching processes with and without immigration. These provide convenient formulations of the processes.

In particular, the result of Theorem 1.4 is derived as an immediate consequence of (1.4) and (1.5). We hope the equations can also be useful in some other similar situations. The proof of Theorem 1.5 is based on Theorem 1.4 and the results of Chen (1997) and Chen et al. (2006).

The study of recurrence of the immigration model is more delicate since the problem cannot be reduced to the extinction problem of the original nonlinear branching process as in the case of a resurrection model. Theorem 1.6 was proved by using the results of the minimal nonnegative solutions as developed in Chen (2004) and comparing the process with some linear branching processes which was studied by Li and Chen (2006).

The proofs of the ergodicities in Theorems 1.7 and 1.8 are based on comparisons of the process with some suitably designed birth-death process and estimates of the mean extinction time.

2 Stochastic integral equations

Stochastic integral equations with jumps have been playing increasingly important roles in the study of Markov processes. In this section, we give a construction of the solution to (1.4) and prove the solution is a minimal nonlinear branching process with immigration. This result is then used to study the regularity of the density matrix Q . We refer to Ikeda and Watanabe (1989) for the general theory of stochastic equations with jumps.

Proposition 2.1 *The pathwise uniqueness of solutions holds for the equation (1.4).*

Proof. Let $\{X_t\}$ and $\{X'_t\}$ be any two solutions of equation (1.4) with $X_0 = X'_0$. By passing to the conditional probability $P(\cdot|\mathcal{F}_0)$, we may and do assume $X_0 = X'_0$ is deterministic. Let $\tau_m = \inf \{t \geq 0 : X_t \geq m\}$, $\tau'_m = \inf \{t \geq 0 : X'_t \geq m\}$ and $\sigma_m = \tau_m \wedge \tau'_m$. It is sufficient to show that $\tau_m = \tau'_m = \sigma_m$ and $X_t = X'_t$ for all $t \leq \sigma_m$ ($m = 1, 2, \dots$). Then

$$X_{t \wedge \sigma_m} - X'_{t \wedge \sigma_m} = \int_0^{t \wedge \sigma_m} \int_0^\infty \int_{\mathbb{N}_{m+1}} (z-1)[1_{\{0 < u \leq r(X_{s-})\}} - 1_{\{0 < u \leq r(X'_{s-})\}}] N_p(ds, du, dz),$$

where $\mathbb{N}_m = \{0, 1, 2, \dots, m\}$. Taking the expectation, we get

$$\begin{aligned} & E[|X_{t \wedge \sigma_m} - X'_{t \wedge \sigma_m}|] \\ & \leq E \left\{ \int_0^{t \wedge \sigma_m} \int_0^\infty \int_{\mathbb{N}_{m+1}} |(z-1)[1_{\{0 < u \leq r(X_{s-})\}} - 1_{\{0 < u \leq r(X'_{s-})\}}]| N_p(ds, du, dz) \right\} \\ & \leq E \left\{ \int_0^{t \wedge \sigma_m} \int_0^\infty \int_{\mathbb{N}_{m+1}} (z+1) |1_{\{0 < u \leq r(X_{s-})\}} - 1_{\{0 < u \leq r(X'_{s-})\}}| ds du m(dz) \right\} \\ & \leq (M_{m+1} + 1) E \left\{ \int_0^{t \wedge \sigma_m} |r(X_{s-}) - r(X'_{s-})| ds \right\} \\ & \leq (M_{m+1} + 1) \int_0^t E[|r(X_{s \wedge \sigma_m-}) - r(X'_{s \wedge \sigma_m-})|] ds, \end{aligned}$$

where $M_m := \int_{\mathbb{N}_m} zm(dz)$. By taking $m \geq X_0$, we have $X_{s-} \vee X'_{s-} \leq m$ for $0 < s \leq \sigma_m$. Denote $d_m = \sup\{|(r(i) - r(j))/(i - j)| : i \neq j, \quad 0 \leq i, j \leq m\}$. Then we have

$$\begin{aligned} E[|X_{t \wedge \sigma_m} - X'_{t \wedge \sigma_m}|] \\ \leq (M_{m+1} + 1)d_m \int_0^t E[|X_{s \wedge \sigma_m} - X'_{s \wedge \sigma_m}|] ds. \end{aligned} \quad (2.1)$$

Since $X_{s \wedge \sigma_m}$ and $X'_{s \wedge \sigma_m}$ only have countably many discontinuous points, we can also use $X_{s \wedge \sigma_m}$ and $X'_{s \wedge \sigma_m}$ instead of $X_{s \wedge \sigma_m-}$ and $X'_{s \wedge \sigma_m-}$ in the right hand side of (2.1). Using Gronwall's inequality we have $E[|X_{s \wedge \sigma_m} - X'_{s \wedge \sigma_m}|] = 0$. Thus we can conclude that $X_t = X'_t$ for all $t \in [0, \sigma_m)$ a.s. This clearly implies that $\tau_m = \tau'_m = \sigma_m$ a.s. and the pathwise uniqueness of solutions of (1.4) is proven.

Theorem 2.2 *For any \mathbb{N} -valued \mathcal{F}_0 -measurable random variable X_0 , there is a pathwise unique solution to (1.5)*

Proof. Without loss of generality, we assume X_0 is deterministic. Let $D_1 = \{s : p(s) \in (0, r(X_0)] \times \mathbb{N}\}$. Since

$$E[N_p((0, t] \times (0, r(X_0)] \times \mathbb{N})] = \int_0^t ds \int_0^{r(X_0)} du \int_{\mathbb{N}} m(dz) = tr(X_0) < \infty,$$

the set D_1 is discrete in $(0, \infty)$. Let σ_1 be the minimal element in D_1 and $p(\sigma_1) = (u_1, z_1)$. Then set

$$X_t = \begin{cases} X_0, & t \in [0, \sigma_1) \\ X_0 + (z_1 - 1), & t = \sigma_1. \end{cases}$$

The process $\{X_t : 0 < t \leq \sigma_1\}$ is clearly the solution of (1.5). Set $D_2 = \{s : p(s + \sigma_1) \in [0, r(x(\sigma_1))] \times \mathbb{N}\}$, σ_2 be the minimal element in D_2 and $p(\sigma_1 + \sigma_2) = (u_2, z_2)$. Define $\{X_t : \sigma_1 < t \leq \sigma_1 + \sigma_2\}$ by

$$X_t = \begin{cases} x(\sigma_1), & t \in (\sigma_1, \sigma_1 + \sigma_2) \\ x(\sigma_1) + (z_2 - 1), & t = \sigma_1 + \sigma_2. \end{cases}$$

It is easy to see that $\{X_t : 0 < t \leq \sigma_1 + \sigma_2\}$ is the unique solution of (1.5). Continuing this process successively, we get a process $\{X_t : 0 \leq t < \tau\}$, where $\tau = \sum_{i=1}^{\infty} \sigma_i$. Next, we show $\tau = \zeta := \lim_{k \rightarrow \infty} \tau_k$, where $\tau_k = \inf\{t \geq 0 : X_t \geq k\}$. Clearly, for each $n \geq 0$ we have $X_t < \infty$ for $t \in [0, \sum_{i=0}^n \sigma_i]$. Then $\sum_{i=0}^n \sigma_i < \zeta$ holds for each $n \geq 0$, and so $\tau \leq \zeta$. On the other hand, since

$$E\left[\int_0^{t \wedge \tau_m} \int_0^{r(X_{s-})} \int_{\mathbb{N}} N_p(ds, du, dz)\right] \leq t \max_{0 \leq k \leq m} r(k) < \infty,$$

the process $\{X_t\}$ has finitely many jumps before $t \wedge \tau_m$, therefore $t \wedge \tau_m < \tau$, since $t \geq 0$ and $m \geq 1$ can be arbitrary, we get $\zeta \leq \tau$. Then we have $\tau = \zeta$. Hence X_t is determined in the time interval $[0, \zeta)$; the uniqueness is clear from Proposition 2.1. \square

Proof of Theorem 1.1. Let $\{X_t^0\}$ denote the solution to (1.5). Let $\{v_k : k = 1, 2, \dots\}$ be the set of jump times of the Poisson process

$$t \mapsto \int_0^t \int_{\mathbb{N}} N_q(ds, dz).$$

We have clearly $v_k \rightarrow \infty$ as $k \rightarrow \infty$. For $0 \leq t < v_1$ set $X_t = X_t^0$. Suppose that X_t has been defined for $0 \leq t < v_k$ and let

$$\xi = X_{v_k-} + \int_{\{v_k\}} \int_{\mathbb{N}} z N_q(ds, dz).$$

Here and in the sequel we make the convention $\infty + \dots = \infty$. By the assumption there is also a solution $\{X_t^k\}$ to

$$X_t = \xi + \int_0^t \int_0^{r(X_{s-})} \int_{\mathbb{N}} (z-1) N_p(v_k + ds, du, dz).$$

Let η_k be the explosion time of $\{X_t^k\}$. If $v_k + \eta_k > v_{k+1}$, we define $X_t = X_{t-v_k}^k$ for $v_k \leq t < v_{k+1}$. If $v_k + \eta_k \leq v_{k+1}$, we set $X_t = X_{t-v_k}^k$ for $v_k \leq t < v_k + \eta_k$ and $X_t = \infty$ for $v_k + \eta_k \leq t < v_{k+1}$. By induction that defines a process $\{X_t\}$, which is clearly the pathwise unique solution to (1.4). Obviously, if the solution of (1.5) is non-explosive for each deterministic initial state $X_0 = i \in \mathbb{N}$, we have $\eta_k = \infty$ for all $k \in \mathbb{N}$, and so $\{X_t\}$ is non-explosive. \square

Proof of Theorem 1.2. Let $\tilde{N}_p(ds, du, dz) = N_p(ds, du, dz) - ds d\mu(dz)$ and $\tilde{N}_q(ds, dz) = N_q(ds, dz) - ds d\mu(dz)$. For any bounded function f on \mathbb{N} we have,

$$\begin{aligned} f(X_{t \wedge \tau_m}) &= f(X_0) + \int_0^{t \wedge \tau_m} \int_0^{r(X_{s-})} \int_{\mathbb{N}} [f(X_{s-} + z - 1) - f(X_{s-})] N_p(ds, du, dz) \\ &\quad + \int_0^{t \wedge \tau_m} \int_{\mathbb{N}} [f(X_{s-} + z) - f(X_{s-})] N_q(ds, dz) \\ &= f(X_0) + \int_0^{t \wedge \tau_m} \int_0^{r(X_{s-})} \int_{\mathbb{N}} [f(X_{s-} + z - 1) - f(X_{s-})] ds d\mu(dz) \\ &\quad + \int_0^{t \wedge \tau_m} \int_{\mathbb{N}} [f(X_{s-} + z) - f(X_{s-})] \gamma ds n(dz) + M_t(f), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M_t(f) &:= \int_0^{t \wedge \tau_m} \int_0^{r(X_{s-})} \int_{\mathbb{N}} [f(X_{s-} + z - 1) - f(X_{s-})] \tilde{N}_p(ds, du, dz) \\ &\quad + \int_0^{t \wedge \tau_m} \int_{\mathbb{N}} [f(X_{s-} + z) - f(X_{s-})] \tilde{N}_q(ds, dz) \end{aligned}$$

is a martingale. Since $X_s \neq X_{s-}$ for at most countably many $s \geq 0$, we can also use X_s instead of X_{s-} in the right hand side of (2.2). In particular, for $f = 1_{\{j\}}$ we have

$$1_{\{X_{t \wedge \tau_m} = j\}} = 1_{\{X_0 = j\}} + \sum_{k=0}^{\infty} b_k \int_0^{t \wedge \tau_m} r(X_s) [1_{\{X_s + k - 1 = j\}} - 1_{\{X_s = j\}}] ds$$

$$+ \sum_{k=1}^{\infty} \gamma a_k \int_0^{t \wedge \tau_m} [1_{\{X_s+k=j\}} - 1_{\{X_s=j\}}] ds + M_t(1_{\{j\}}).$$

Write $E_i = E(\cdot | X_0 = i)$ for $i \in \mathbb{N}$. Taking the expectation in both sides of the above equation and letting $m \rightarrow \infty$ we get

$$\begin{aligned} E_i(1_{\{X_{t \wedge \zeta}=j\}}) &= E_i(1_{\{X_0=j\}}) + \sum_{k=0}^{\infty} b_k E_i \left(\int_0^{t \wedge \zeta} r(X_s) [1_{\{X_s+k-1=j\}} - 1_{\{X_s=j\}}] ds \right) \\ &+ \sum_{k=1}^{\infty} \gamma a_k E_i \left(\int_0^{t \wedge \zeta} [1_{\{X_s+k=j\}} - 1_{\{X_s=j\}}] ds \right). \end{aligned}$$

Obviously, here we can remove the truncation “ $\wedge \zeta$ ” and obtain

$$\begin{aligned} Q_{ij}(t) &= \delta_{ij} + \sum_{k=0}^j b_k \int_0^t [r_{j-k+1} Q_{i,j-k+1}(s) - r_j Q_{ij}(s)] ds \\ &+ \sum_{k=1}^j \int_0^t \gamma a_k [Q_{i,j-k}(s) - Q_{ij}(s)] ds \\ &= \delta_{ij} + \int_0^t \left(\sum_{k=1}^{j+1} Q_{ik}(s) r_k b_{j-k+1} - Q_{ij}(s) r_j \right) ds \\ &+ \int_0^t \left(\sum_{k=0}^{j-1} Q_{ik}(s) \gamma a_{j-k} - \gamma Q_{ij}(s) \right) ds. \end{aligned}$$

Differentiating both sides we get

$$\begin{aligned} Q'_{ij}(t) &= \sum_{k=1}^{j+1} Q_{ik}(t) r_k b_{j-k+1} - Q_{ij}(t) r_j \\ &+ \sum_{k=0}^{j-1} Q_{ik}(t) \gamma a_{j-k} - \gamma Q_{ij}(t) \\ &= \sum_{k=0}^{\infty} Q_{ik}(t) q_{kj}. \end{aligned}$$

This is just the Kolmogorov forward equation of Q . □

Proof of Theorem 1.3. By Theorem 1.1, the solution $\{X_t\}$ to (1.4) is a time homogeneous Markov process with state space $\bar{\mathbb{N}} := \{0, 1, 2, \dots, \infty\}$. Suppose that σ_1 and z_1 are given in the proof of Theorem 2.2. Let $q(v_1) = y_1$. By the properties of Poisson point process, we can see that $P(\sigma_1 > t) = e^{-r(X_0)t}$, $P(z_1 = i) = m(\{i\}) = b_i$, $P(v_1 > t) = e^{-\gamma t}$, $P(y_1 = i) = n(\{i\}) = a_i$ and σ_1, z_1, v_1, y_1 are mutually independent. Write $P_i = P(\cdot | X_0 = i)$ for $i \in \mathbb{N}$. Let $\xi_t = \max\{n + m : \sum_{i=0}^n \sigma_i, \sum_{i=0}^m v_i \leq t\}$. Obviously we have $P_i[X_t = j, \xi_t = 0] = \delta_{ij}$. By the Markov property of $\{X_t\}$,

$$P_i\{X_t = j, \xi_t = m + 1\} = P_i\left\{1_{\{\sigma_1 \wedge v_1 < t\}} P_{X_{\sigma_1 \wedge v_1}}[X_{t-\sigma_1 \wedge v_1} = j, \xi_{t-\sigma_1 \wedge v_1} = m]\right\}$$

$$\begin{aligned}
&= P_i \left\{ 1_{\{\sigma_1 < t\}} 1_{\{v_1 \geq \sigma_1\}} P_{X_{\sigma_1}} \left[X_{t-\sigma_1} = j, \xi_{t-\sigma_1} = m \right] \right\} \\
&\quad + P_i \left\{ 1_{\{v_1 < t\}} 1_{\{v_1 < \sigma_1\}} P_{X_{v_1}} \left[X_{t-v_1} = j, \xi_{t-v_1} = m \right] \right\} \\
&= P_i \left\{ \int_0^t r_i e^{-r_i(t-s)} e^{-\gamma(t-s)} P_{X_{t-s}} [X_s = j, \xi_s = m] ds \right\} \\
&\quad + P_i \left\{ \int_0^t \gamma e^{-\gamma(t-s)} e^{-r_i(t-s)} P_{X_{t-s}} [X_s = j, \xi_s = m] ds \right\} \\
&= P_i \left\{ \int_0^t r_i e^{-(r_i+\gamma)(t-s)} \sum_{k=i-1}^{\infty} P(z_1 = k - i + 1) P_k [X_s = j, \xi_s = m] ds \right\} \\
&\quad + P_i \left\{ \int_0^t \gamma e^{-(r_i+\gamma)(t-s)} \sum_{k=i+1}^{\infty} P(y_1 = k - i) P_k [X_s = j, \xi_s = m] ds \right\} \\
&= \sum_{k \neq i} \int_0^t e^{-(r_i+\gamma)(t-s)} q_{ik} P_k [X_s = j, \xi_s = m] ds.
\end{aligned}$$

Notice that

$$P_i[X_t = j] = \sum_{m=0}^{\infty} P_i[X_t = j, \xi_t = m].$$

From the theory of Markov chains we know $P_{ij}(t) := P_i[X_t = j]$ is the minimal solution to the Kolmogorov equation of the density matrix Q , see Chen (2004, p.78). Then $\{X_t\}$ is the minimal process of the density matrix Q . \square

Proof of Theorem 1.4. Suppose that R is regular. Then the minimal solution of its Kolmogorov backward equation is honest i.e. the minimal process of R is non-explosive. Applying Theorem 1.1 and 1.3 we know the minimal process of Q is non-explosive. Thus Q is regular. Conversely, suppose that R is not regular. Then by Theorem 2.7 (3) in Anderson (1991, p.80) there exists a non-trivial solution (u_i^*) to

$$u_i \leq \sum_{k \neq i} \frac{r_{ik}}{2\gamma + r_i} u_k, \quad 0 \leq u_i \leq 1.$$

Since $r_{ik} \leq q_{ik}$, we see (u_i^*) is also a solution to

$$u_i \leq \sum_{k \neq i} \frac{q_{ik}}{\gamma + q_i} u_k.$$

Using Theorem 2.7 (3) in Anderson (1991, p.80) again, we see Q is not regular. \square

Proof of Theorem 1.5. By Theorem 1.4 we derive the results from Theorem 1.2 of Chen (1997) and Theorem 2.3 of Chen et al. (2006). \square

3 Recurrence

Proof of Theorem 1.6. (1) Under the assumption, there exists a constant $N \geq 1$ such that $r_i \geq \gamma m / (1 - M)$ holds for each $i \geq N$. Take $x_i = i$ for $i \geq 0$. For $i \geq N$ we have

$$\begin{aligned} \sum_{j=0}^{\infty} q_{ij} x_j &= r_i b_0 (i - 1) + \sum_{j=1}^{\infty} (r_i b_{j+1} + \gamma a_j) (i + j) \\ &= (r_i + \gamma) i + r_i (M - 1) + \gamma m \leq (r_i + \gamma) i = -q_{ii} x_i. \end{aligned}$$

Let (π_{ij}) be the embedded chain of (q_{ij}) . The above calculations imply that (x_i) is a finite solution of

$$\sum_{j=0}^{\infty} \pi_{ij} x_j \leq x_i, \quad i \geq N.$$

Then Q is recurrent by Theorem 4.24 in Chen (2004, p.134).

(2) Suppose that $M \leq 1$ and $J = \infty$. We shall prove the process is recurrent by comparison arguments. Let $\bar{Q} = (\bar{q}_{ij})$ be the density matrix defined by

$$\bar{q}_{ij} = \begin{cases} \alpha i b_{j-i+1} + \gamma a_{j-i} & j \geq i + 1 \\ -\alpha i - \gamma & j = i \\ \alpha i b_0 & j = i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

which corresponds to a linear branching process with immigration. It was proved in Li and Chen (2006) that this process is recurrent. Next, we define the density matrix $Q^* = (q_{ij}^*)$ by

$$q_{ij}^* = \begin{cases} r_i b_{j-i+1} + \gamma a_{j-i} r_i / \alpha i & j \geq i + 1 \\ -r_i - \gamma r_i / \alpha i & j = i \\ r_i b_0 & j = i - 1 \\ q_{ij} & i < N \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\bar{\pi}_{ij})$ and (π_{ij}^*) denote the embedded chains of (\bar{q}_{ij}) and (q_{ij}^*) , respectively. It is easy to see that $\bar{\pi}_{ij} = \pi_{ij}^*$ for $i \geq N$ and $j \geq 0$. Then Q^* is also recurrent. For $l \geq i > N$ we have

$$\sum_{j=i}^{\infty} q_{ij} = -r_i b_0 \leq \sum_{j=i}^{\infty} q_{lj}^*.$$

Moreover, we have

$$\sum_{j=k}^{\infty} q_{ij} = \sum_{j=k}^{\infty} q_{lj}^* = 0, \quad k \leq i - 1$$

and

$$\sum_{j=k}^{\infty} q_{ij} \leq \sum_{j=k}^{\infty} q_{ij}^* \leq \sum_{j=k}^{\infty} q_{lj}^*, \quad k \geq l+1.$$

Then Q and Q^* are stochastically comparable, so we can construct a Q -process (X_t) and a Q^* -process (X_t^*) on some probability space in such a way that $X_0 = X_0^*$ and $X_t \leq X_t^*$ for all $t \geq 0$; see Example 5.51 in Chen (2004, p.220). Now the recurrence of (X_t) follows from that of (X_t^*) .

(3) Since $M > 1$, there exists a $s \in (0, 1)$ such that $B(s) < 0$ i.e. $\sum_{i=0}^{\infty} b_i s^{i-1} < 1$. Take $H = \{0\}$ and $x_i = 1 - s^i$. For $i \geq 1$ we have

$$\begin{aligned} \sum_{k=0}^{\infty} \pi_{ik} x_k &= \pi_{i,i-1} x_{i-1} + \sum_{k=1}^{\infty} \pi_{i,i+k} x_{i+k} \\ &= \frac{r_i b_0}{r_i + \gamma} x_{i-1} + \sum_{k=1}^{\infty} \frac{r_i b_{k+1} + \gamma a_k}{r_i + \gamma} x_{i+k} \\ &= \frac{1}{r_i + \gamma} \left[r_i b_0 (1 - s^{i-1}) + \sum_{k=1}^{\infty} \gamma a_k (1 - s^{i+k}) + \sum_{k=1}^{\infty} r_i b_{k+1} (1 - s^{i+k}) \right] \\ &= 1 - \frac{s^i}{r_i + \gamma} \left[r_i \sum_{k=0}^{\infty} b_k s^{k-1} + \gamma \sum_{k=1}^{\infty} a_k s^k \right] \geq 1 - s^i = x_i. \end{aligned}$$

Then the process is transient by Theorem 8.0.2 in Meyn and Tweedie (2009).

(4) Since the proof is similar to the proof of (2), we omit it. \square

4 Mean extinction time

In this section, we assume $r_i = \alpha i^\theta$ for $\alpha > 0$ and $\theta \geq 1$. Let (X_t) be a realization of the nonlinear branching process with immigration. Its jump times are given successively by $\tau_0 = 0$ and $\tau_n = \inf\{t : t > \tau_{n-1}, X_t \neq X_{\tau_{n-1}}\}$. We also define $\sigma_k = \inf\{t \geq \tau_1 : X_t = k\}$. In order to prove the criterion for the ergodicity of (X_t) , let us consider the absorbing process $\tilde{X}_t := X_{t \wedge \sigma_0}$. The density matrix of this process is given by:

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & i \neq 0 \\ 0 & i = 0. \end{cases}$$

For this process, we define $\tilde{\tau}_0 = 0$, $\tilde{\tau}_n = \inf\{t : t > \tilde{\tau}_{n-1}, \tilde{X}(t) \neq \tilde{X}(\tilde{\tau}_{n-1})\}$ and $\tilde{\sigma}_k = \inf\{t \geq \tau_1 : \tilde{X}_t = k\}$. It is easy to see that

$$E_i \sigma_0 = E_i \tilde{\sigma}_0. \quad (4.1)$$

Let $(\tilde{p}_{ij}(t))$ and $(\tilde{\phi}_{ij}(\lambda))$ denote the transition function and the resolvent of (\tilde{X}_t) , respectively.

Lemma 4.1 For any $i \geq 0$ and $s \in [0, 1)$, we have

$$\sum_{j=0}^{\infty} \tilde{p}'_{ij}(t) s^j = \alpha B(s) \sum_{j=1}^{\infty} \tilde{p}_{ij}(t) j^{\theta} s^{j-1} - A(s) \sum_{j=1}^{\infty} \tilde{p}_{ij}(t) s^j, \quad t \geq 0, \quad (4.2)$$

and

$$\lambda \sum_{j=0}^{\infty} \tilde{\phi}'_{ij}(\lambda) s^j - s^i = \alpha B(s) \sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) j^{\theta} s^{j-1} - A(s) \sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) s^j, \quad \lambda > 0. \quad (4.3)$$

Proof. From the Kolmogorov forward equation of the transition function we obtain that

$$\tilde{p}'_{ij}(t) = \sum_{k=1}^{j-1} \tilde{p}_{ik}(t) (r_k b_{j-k+1} + \gamma a_{j-k}) - \tilde{p}_{ij}(t) (r_j + \gamma) + \tilde{p}_{i,j+1}(t) r_{j+1} b_0.$$

Multiplying s^j on both sides of the above equality and then summing over j , we have

$$\begin{aligned} \sum_{j=0}^{\infty} \tilde{p}'_{ij}(t) s^j &= \sum_{j=0}^{\infty} \sum_{k=1}^{j-1} \tilde{p}_{ik}(t) r_k b_{j-k+1} s^j + \gamma \sum_{j=0}^{\infty} \sum_{k=1}^{j-1} \tilde{p}_{ik}(t) a_{j-k} s^j \\ &\quad + \sum_{j=0}^{\infty} \tilde{p}_{i,j+1}(t) r_{j+1} s^j b_0 - \sum_{j=1}^{\infty} \tilde{p}_{ij}(t) r_j s^j - \gamma \sum_{j=0}^{\infty} \tilde{p}_{ij}(t) s^j, \end{aligned}$$

Then we can interchange the order of summation to see

$$\sum_{j=0}^{\infty} \sum_{k=0}^{j-1} \tilde{p}_{ik}(t) r_k b_{j-k+1} s^j = \sum_{k \neq l} \tilde{p}_{ik}(t) r_k s^{k-1} \sum_{j=k+1}^{\infty} b_{j-k+1} s^{j-k+1}$$

and

$$\gamma \sum_{j=0}^{\infty} \sum_{k=0}^{j-1} \tilde{p}_{ik}(t) a_{j-k} s^j = \gamma \sum_{k \neq l} \tilde{p}_{ik}(t) s^k \sum_{j=k+1}^{\infty} a_{j-k} s^{j-k}.$$

It follows that

$$\sum_{j=0}^{\infty} \tilde{p}'_{ij}(t) s^j = \sum_{j=1}^{\infty} \tilde{p}_{ij}(t) r_j s^{j-1} \alpha B(s) - \sum_{j=0}^{\infty} \tilde{p}_{ij}(t) s^j A(s).$$

That proves (4.2) and (4.3) is just the Laplace transform of (4.2). \square

Lemma 4.2 For any $i, k \geq 1$, we have $\int_0^{\infty} \tilde{p}_{ik}(t) dt < \infty$ and $\lim_{t \rightarrow \infty} \tilde{p}_{ik}(t) = 0$. Furthermore, for $i \geq 1$ and $s \in [0, 1)$, we have

$$\sum_{k=1}^{\infty} \left(\int_0^{\infty} \tilde{p}_{ik}(t) dt \right) s^k < \infty. \quad (4.4)$$

Proof. Fixing an $i \geq 1$, we can use the Kolmogorov forward equation to see

$$\tilde{p}_{i0}(t) = b_0 \alpha \int_0^t \tilde{p}_{i1}(u) du,$$

which means that

$$\int_0^\infty \tilde{p}_{i1}(t) dt \leq b_0^{-1} \alpha^{-1} < \infty.$$

Suppose that $\int_0^\infty \tilde{p}_{ik}(t) dt < \infty$ for $k \leq j$. By the Kolmogorov forward equations we can see for $j \geq 1$,

$$\begin{aligned} \tilde{p}_{ij}(t) - \delta_{ij} &= \sum_{k=1}^{j-1} (\alpha k^\theta b_{j-k+1} + \gamma a_{j-k}) \int_0^t \tilde{p}_{ik}(u) du - (\alpha j^\theta + \gamma) \int_0^t \tilde{p}_{ij}(u) du \\ &\quad + \alpha(j+1)^\theta b_0 \int_0^t \tilde{p}_{ij+1}(u) du. \end{aligned}$$

Letting $t \rightarrow \infty$, we have

$$\int_0^\infty \tilde{p}_{ij+1}(t) dt < \infty.$$

Then $\int_0^\infty \tilde{p}_{ik}(t) dt < \infty$ by induction. Since the limit $\lim_{t \rightarrow \infty} \tilde{p}_{ik}(t)$ always exists, we see $\lim_{t \rightarrow \infty} \tilde{p}_{ik}(t) = 0$ immediately.

We next tend to prove (4.4). Since $M \leq 1$, we have $B(s) > 0$ for a fixed $s \in [0, 1]$. Then there exists a $k \geq 1$ so that $k\alpha B(s) - sA(s) > 0$. Using (4.2), we have

$$\begin{aligned} \sum_{j=0}^\infty \tilde{p}'_{ij}(u) s^j &= \alpha B(s) \sum_{j=1}^\infty \tilde{p}_{ij}(u) j^\theta s^{j-1} - A(s) \sum_{j=1}^\infty \tilde{p}_{ij}(u) s^j \\ &\geq \alpha B(s) \sum_{j=k+1}^\infty \tilde{p}_{ij}(u) j^\theta s^{j-1} - A(s) \sum_{j=1}^\infty \tilde{p}_{ij}(u) s^j \\ &\geq [k\alpha B(s) - sA(s)] \sum_{j=k+1}^\infty \tilde{p}_{ij}(u) s^{j-1} - A(s) \sum_{j=1}^k \tilde{p}_{ij}(u) s^j. \end{aligned} \quad (4.5)$$

Let $\|A\| = \max_{s \in [0, 1]} |A(s)|$ and $\|B\| = \max_{s \in [0, 1]} |\alpha B(s)|$. Then for each $s \in [0, 1]$,

$$\begin{aligned} \int_0^t \sum_{j=0}^\infty |\tilde{p}'_{ij}(u) s^j| du &\leq \|B\| \int_0^t \sum_{j=1}^\infty \tilde{p}_{ij}(u) j^\theta s^{j-1} du + \|A\| \int_0^t \sum_{j=1}^\infty \tilde{p}_{ij}(u) s^j du \\ &\leq t\|B\| \sum_{j=1}^\infty j^\theta s^{j-1} + t\|A\| \sum_{j=1}^\infty s^j < \infty. \end{aligned}$$

Then we use Fubini's theorem to see

$$\int_0^t \sum_{j=0}^\infty \tilde{p}'_{ij}(u) s^j du = \sum_{j=0}^\infty \int_0^t \tilde{p}'_{ij}(u) s^j du.$$

Integrating both sides of (4.5),

$$\begin{aligned} \sum_{j=0}^{\infty} \tilde{p}_{ij}(t) s^j - s^i &\geq [k\alpha B(s) - sA(s)] \cdot \sum_{j=k+1}^{\infty} \left(\int_0^t \tilde{p}_{ij}(u) du \right) s^{j-1} \\ &\quad - A(s) \cdot \sum_{j=1}^k \left(\int_0^t \tilde{p}_{ij}(u) du \right) s^j. \end{aligned}$$

Letting $t \rightarrow \infty$ and using the fact that $\int_0^{\infty} \tilde{p}_{ik}(t) dt < \infty$, we have

$$\sum_{j=k+1}^{\infty} \left(\int_0^{\infty} \tilde{p}_{ij}(u) du \right) s^{j-1} < \infty,$$

which implies (4.4). □

Proposition 4.3 *Suppose that the nonlinear branching process with immigration is recurrent and (1.6) holds. Then for $i \geq 1$ we have*

$$E_i(\sigma_0) \leq \frac{1}{\Gamma(\theta)} \int_0^1 \frac{1-y^i}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \cdot \exp \left[\frac{1}{\Gamma(\theta)} \int_0^1 \frac{A(y)}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \right]. \quad (4.6)$$

and

$$E_i(\sigma_0) \geq \int_0^1 \frac{1-y^i}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy. \quad (4.7)$$

Proof. Multiplying (4.3) by $(\ln(s/y))^{\theta-1}$, dividing by $\alpha B(s)$ and integrating both sides we have

$$\int_0^s \sum_{j=1}^{\infty} \tilde{\phi}_{ij} j^{\theta} y^{j-1} \left(\ln \frac{s}{y} \right)^{\theta-1} dy = \int_0^s \frac{(\lambda + A(y)) \sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) y^j - y^i + \lambda \tilde{\phi}_{i0}(\lambda)}{\alpha B(y)} \left(\ln \frac{s}{y} \right)^{\theta-1} dy.$$

Letting $y = se^{-\frac{x}{\Gamma(\theta)}}$ in the left hand side of the above equation we get

$$\int_0^s \sum_{j=1}^{\infty} \tilde{\phi}_{ij} j^{\theta} y^{j-1} \left(\ln \frac{s}{y} \right)^{\theta-1} dy = \int_0^{\infty} \sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) s^j x^{\theta-1} e^{-x} dx = \Gamma(\theta) \sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) s^j.$$

Using the above two equations we obtain

$$\sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) s^j = \frac{1}{\Gamma(\theta)} \int_0^s \frac{(\lambda + A(y)) \sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) y^j - y^i + \lambda \tilde{\phi}_{i0}(\lambda)}{\alpha B(y)} \left(\ln \frac{s}{y} \right)^{\theta-1} dy. \quad (4.8)$$

For $i \geq 1$, $\lambda > 0$ and $s \in [0, 1]$ let

$$\psi_i(\lambda, s) = \sum_{j=1}^{\infty} \tilde{\phi}_{ij}(\lambda) s^j.$$

Note that

$$\lambda \tilde{\phi}_{i0}(\lambda) = \int_0^\infty e^{-t} p_{i0}\left(\frac{t}{\lambda}\right) dt \leq \int_0^\infty e^{-t} dt = 1.$$

Then, by (4.8),

$$\begin{aligned} \psi_i(\lambda, s) &\leq \frac{1}{\Gamma(\theta)} \int_0^1 \frac{1-y^i}{\alpha B(y)} \left(\ln \frac{1}{y}\right)^{\theta-1} dy \\ &\quad + \frac{1}{\Gamma(\theta)} \int_0^s \frac{(\lambda + A(y))\psi_i(\lambda, y)}{\alpha B(y)} \left(\ln \frac{1}{y}\right)^{\theta-1} dy. \end{aligned} \quad (4.9)$$

By Lemma 4.2,

$$\lim_{\lambda \rightarrow 0} \lambda \sum_{j=1}^\infty \tilde{\phi}_{ij}(\lambda) = \lim_{\lambda \rightarrow 0} \sum_{j=1}^\infty \int_0^\infty e^{-t} \tilde{p}_{ij}\left(\frac{t}{\lambda}\right) dt = 0.$$

It follows that, for $s \in [0, 1]$,

$$\lim_{\lambda \rightarrow 0} \lambda \psi_i(\lambda, s) \leq \lim_{\lambda \rightarrow 0} \lambda \sum_{j=1}^\infty \tilde{\phi}_{ij}(\lambda) = 0. \quad (4.10)$$

Denote

$$\begin{aligned} C_i &:= \frac{1}{\Gamma(\theta)} \int_0^1 \frac{1-y^i}{\alpha B(y)} \left(\ln \frac{1}{y}\right)^{\theta-1} dy \\ &\leq \frac{1}{\Gamma(\theta)} \int_0^1 \frac{1-y}{\alpha B(y)} \left(\ln \frac{1}{y}\right)^{\theta-1} dy \\ &\leq \frac{1}{\Gamma(\theta)} \int_0^1 \frac{A(y)}{\alpha B(y)} \left(\ln \frac{1}{y}\right)^{\theta-1} dy < \infty. \end{aligned}$$

By (4.4) we have

$$\psi_i(0, s) = \sum_{k=1}^\infty \left(\int_0^\infty p_{ik}(t) dt \right) s^k < \infty$$

for each $0 \leq s < 1$. Letting $\lambda \rightarrow 0$ in (4.9), we have

$$\psi_i(0, s) \leq C_i + \frac{1}{\Gamma(\theta)} \int_0^s \frac{A(y)\psi_i(0, y)}{\alpha B(y)} \left(\ln \frac{1}{y}\right)^{\theta-1} dy.$$

Using the Gronwall's inequality, we have

$$\psi_i(0, s) \leq C_i \exp \left[\frac{1}{\Gamma(\theta)} \int_0^s \frac{A(y)}{\alpha B(y)} \left(\ln \frac{1}{y}\right)^{\theta-1} dy \right]. \quad (4.11)$$

Letting $s \uparrow 1$ we see

$$\lim_{s \uparrow 1} \psi_i(0, s) = \lim_{s \uparrow 1} \sum_{j=1}^\infty \int_0^\infty \tilde{p}_{ij}(t) s^j dt = \int_0^\infty (1 - \tilde{p}_{i0}(t)) dt = E_i(\tilde{\sigma}_0).$$

Hence (4.6) follows from (4.1) and (4.11).

Similarly, by (4.8) we have

$$\psi_i(\lambda, s) \geq \frac{1}{\Gamma(\theta)} \int_0^s \frac{\lambda \tilde{\phi}_{i0}(\lambda) - y^i}{\alpha B(y)} \left(\ln \frac{s}{y} \right)^{\theta-1} dy.$$

Letting $\lambda \rightarrow 0$ and then letting $s \rightarrow 1$, we obtain (4.7). \square

5 Ergodicity and strong ergodicity

One of the main steps to prove Theorems 1.7 and 1.8 is to compare our nonlinear branching process with immigration with a suitably designed birth-death process, which we now introduce. A similar birth-death process was used by Chen (1997) in her study of the regularity of the nonlinear branching process with resurrection. Let

$$L = M + b_0 - 1 = \sum_{k=1}^{\infty} k b_{k+1}$$

and let (\hat{X}_t) be a birth-death process with birth rate $d_i = r_i L + \gamma m$ and death rate $c_i = r_i b_0$. We denote the density matrix of (\hat{X}_t) by (\hat{q}_{ij}) . Let $T_0 := \inf\{t \geq 0 : \hat{X}_t = 0\}$.

Lemma 5.1 (1) Suppose that $m < \infty$, $M < 1$, r_i is increasing and $\sum_{i=1}^{\infty} r_i^{-1} < \infty$. Then the birth-death process (\hat{X}_t) is strongly ergodic.

(2) Suppose that $m < \infty$, $M \leq 1$, r_i is non-decreasing and $\sum_{i=1}^{\infty} r_i^{-1} < \infty$. Then the birth-death process (\hat{X}_t) is ergodic.

Proof. (1) It is easy to check that the birth-death process is regular. Fix an $\varepsilon > 0$ satisfying $L + \varepsilon < b_0$. Then there exists an N such that $d_i \leq \gamma_i(L + \varepsilon)$ for each $i > N$. Let

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{c_{n+1}} + \sum_{k=1}^n \frac{d_k \cdots d_n}{c_k \cdots c_{n+1}} \right). \quad (5.1)$$

It is obvious that $\sum_{n=1}^{\infty} c_{n+1}^{-1} < \infty$. Notice that for each $n > N$ we have

$$\begin{aligned} \sum_{k=1}^n \frac{d_k \cdots d_n}{c_k \cdots c_{n+1}} &\leq \max_{1 \leq k \leq N} \frac{d_k \cdots d_N}{c_k \cdots c_N} \cdot N \cdot \frac{d_{N+1} \cdots d_n}{c_{N+1} \cdots c_{n+1}} + \sum_{k=1}^{n-N} \frac{d_{N+k} \cdots d_n}{c_{N+k} \cdots c_{n+1}} \\ &\leq N \rho^{n-N} \max_{1 \leq k \leq N} \frac{d_k \cdots d_N}{c_k \cdots c_N} + \sum_{k=1}^{n-N} \frac{\rho^{n-N-k+1}}{c_{n+1}}, \end{aligned}$$

where $\rho = b_0^{-1}(L + \varepsilon) < 1$. Then $S < \infty$. By Corollary 2.4 of Zhang (2001), we conclude that (\hat{X}_t) is strongly ergodic.

(2) Since $L \leq b_0$, we have

$$R := \sum_{n=1}^{\infty} \frac{d_0 \cdots d_{n-1}}{c_1 \cdots c_n} \leq d_0 \sum_{n=1}^{\infty} \frac{(c_1 + \gamma m)(c_2 + \gamma m) \cdots (c_{n-1} + \gamma m)}{c_1 c_2 \cdots c_n}.$$

Taking logarithm on the right-hand side we get

$$\ln \left(\frac{(c_1 + \gamma m)(c_2 + \gamma m) \cdots (c_{n-1} + \gamma m)}{c_1 c_2 \cdots c_n} \right) = \sum_{i=1}^{n-1} \ln \left(1 + \frac{\gamma m}{r_i b_0} \right) + \ln \left(\frac{1}{c_n} \right).$$

Since $\lim_{i \rightarrow \infty} r_i = \infty$, we have $\ln \left(1 + \frac{\gamma m}{r_i b_0} \right) \sim \frac{\gamma m}{r_i b_0}$ as $i \rightarrow \infty$. Then there exists a constant $C \geq 0$ such that for sufficiently large n ,

$$\ln \left(\frac{(c_1 + \gamma m)(c_2 + \gamma m) \cdots (c_{n-1} + \gamma m)}{c_1 c_2 \cdots c_n} \right) \leq C \sum_{i=1}^{\infty} \frac{1}{r_i} + \ln \left(\frac{1}{c_n} \right),$$

and hence

$$\frac{(c_1 + \gamma m)(c_2 + \gamma m) \cdots (c_{n-1} + \gamma m)}{c_1 c_2 \cdots c_n} \leq \frac{T}{c_n}$$

for another constant $T \geq 0$. That implies $R < \infty$. By Theorem 4.55 in Chen (2004, p.160) the birth-death process is ergodic. \square

Lemma 5.2 *If the nonlinear branching process with immigration has a stationary distribution $\mu = (\mu_j)$, then the generating function $f(s) := \sum_{j=0}^{\infty} \mu_j s^j$ satisfies the following equation:*

$$\Gamma(\theta) f(s) = \Gamma(\theta) \mu_0 + \int_0^s \frac{A(y)}{\alpha B(y)} \left(\ln \frac{s}{y} \right)^{\theta-1} f(y) dy, \quad s \in [0, 1]. \quad (5.2)$$

Proof. The stationary distribution (μ_j) satisfies $\mu Q = 0$. In view of (1.3), we have

$$\mu_j(\gamma + \alpha j^\theta) = \sum_{i=0}^{j-1} \mu_i \gamma a_{j-i} + \sum_{i=0}^{j+1} \mu_i \alpha i^\theta b_{j-i+1}. \quad (5.3)$$

Multiplying s^j on both sides of the above equality and then summing over j , we have

$$\gamma \sum_{j=1}^{\infty} \mu_j s^j + \alpha s \sum_{j=1}^{\infty} \mu_j j^\theta s^{j-1} = \gamma \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \mu_i a_{j-i} s^j + \alpha \sum_{j=1}^{\infty} \sum_{i=1}^{j+1} \mu_i i^\theta b_{j-i+1} s^j. \quad (5.4)$$

Interchanging the order of summation,

$$\begin{aligned} \text{l.h.s. of (5.4)} &= \gamma \sum_{i=0}^{\infty} \mu_i s^i \sum_{j=i+1}^{\infty} a_{j-i} s^{j-i} - \alpha \mu_1 b_0 \\ &\quad + \alpha \sum_{i=1}^{\infty} \mu_i i^\theta s^{i-1} \sum_{j=i-1}^{\infty} b_{j-i+1} s^{j-i+1} \end{aligned}$$

$$= \gamma \sum_{i=0}^{\infty} \mu_i s^i F(s) - \alpha \mu_1 b_0 + \alpha \sum_{i=1}^{\infty} \mu_i i^{\theta} s^{i-1} G(s). \quad (5.5)$$

Letting $j = 0$ in (5.3), we see $\mu_0 \gamma = \alpha \mu_1 b_0$. Therefore, from (5.4) it follows that

$$\sum_{j=1}^{\infty} \mu_j j^{\theta} s^{j-1} = \frac{f(s)A(s)}{\alpha B(s)}.$$

Multiplying the above equation by $(\ln \frac{s}{y})^{\theta-1}$ and integrating the both sides, we have

$$\int_0^s \sum_{j=1}^{\infty} \mu_j j^{\theta} y^{j-1} \left(\ln \frac{s}{y} \right)^{\theta-1} dy = \int_0^s \frac{A(y)}{\alpha B(y)} \left(\ln \frac{s}{y} \right)^{\theta-1} f(y) dy. \quad (5.6)$$

Letting $y = se^{-\frac{x}{j}}$ we get

$$\begin{aligned} \text{l.h.s. of (5.6)} &= - \int_0^{\infty} \sum_{j=1}^{\infty} \mu_j j^{\theta} (se^{-\frac{x}{j}})^{j-1} \left(\frac{x}{j} \right)^{\theta-1} \left(-\frac{s}{j} e^{-\frac{x}{j}} \right) dx \\ &= \int_0^{\infty} \sum_{j=1}^{\infty} \mu_j s^j x^{\theta-1} e^{-x} dx = \Gamma(\theta) [f(s) - \mu_0]. \end{aligned}$$

Then $f(s)$ is a solution to the differential equation (5.2). \square

Proof of Theorem 1.7. (1) By Lemma 5.1 the birth-death process (\hat{X}_t) is ergodic. Thus by Theorem 4.45 in Chen (2004), the equation

$$u_0 = 0, \quad d_i(u_{i+1} - u_i) + c_i(u_{i-1} - u_i) + 1 = 0, \quad i \neq 0 \quad (5.7)$$

has a finite nonnegative solution (u_i) . By Remark 2.5 of Zhang (2001), we have

$$u_0 = 0, \quad u_i = \sum_{k=0}^{i-1} \left(\frac{1}{c_{k+1}} + \sum_{j=k+1}^{\infty} \frac{d_{k+1} \cdots d_j}{c_{k+1} \cdots c_{j+1}} \right). \quad (5.8)$$

It is apparent that $u_i \leq u_{i+1}$. Moreover, we have

$$u_{i+1} - u_i = \frac{1}{c_{i+1}} + \sum_{j=i+1}^{\infty} \frac{d_{i+1} \cdots d_j}{c_{i+1} \cdots c_{j+1}}, \quad u_i - u_{i-1} = \frac{1}{c_i} + \sum_{j=i}^{\infty} \frac{d_i \cdots d_j}{c_i \cdots c_{j+1}}.$$

Since $d_{i+1}/c_{i+1} < d_i/c_i$ and $1/c_{i+1} < 1/c_i$, it is not hard to show that $u_{i+1} - u_i$ is non-increasing in $i \geq 0$. Coming back to the matrix Q , for $i \geq 1$,

$$\begin{aligned} \sum_{j=0}^{\infty} q_{ij} u_j &= \sum_{j=0}^{\infty} q_{ij} (u_j - u_i) \\ &= c_i(u_{i-1} - u_i) + r_i \sum_{k=1}^{\infty} b_{k+1} \sum_{l=1}^k (u_{i+l} - u_{i+l-1}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} \gamma a_k \sum_{l=1}^k (u_{i+l} - u_{i+l-1}) \\
& \leq c_i(u_{i-1} - u_i) + r_i \sum_{k=1}^{\infty} k b_{k+1} (u_{i+1} - u_i) + \gamma \sum_{k=1}^{\infty} k a_k (u_{i+1} - u_i) \\
& = c_i(u_{i-1} - u_i) + (r_i L + \gamma m)(u_{i+1} - u_i) \\
& = c_i(u_{i-1} - u_i) + d_i(u_{i+1} - u_i) = -1.
\end{aligned} \tag{5.9}$$

and

$$\sum_{j=1}^{\infty} q_{0j} u_j = \sum_{j=1}^{\infty} q_{0j} (u_j - u_1) = \sum_{j=1}^{\infty} q_{0j} \sum_{i=1}^j (u_i - u_{i-1}) < \sum_{j=1}^{\infty} q_{0j} j u_1 \leq \gamma m u_1 < \infty. \tag{5.10}$$

Then (u_i) is a nonnegative bounded solution to the following equation

$$\sum_{j=1}^{\infty} q_{0j} u_j < \infty, \quad \sum_{j=0}^{\infty} q_{ij} u_j \leq -1, \quad i \geq 1.$$

By Theorem 4.45 in Chen (2004, p.145) we know the process is positive recurrent.

(2) Suppose that the process is ergodic. Then letting $s = 1$ in (5.2) we get

$$\infty > \Gamma(\theta) \sum_{j=1}^{\infty} \mu_j \geq \int_0^1 \frac{\sum_{j=0}^{\infty} \mu_j y^j A(y)}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \geq \int_0^1 \frac{\mu_0 A(y)}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy.$$

Since $\mu_0 > 0$, we have (1.6). Conversely, suppose that (1.6) holds. By the strong Markov property, we have

$$E_0(\sigma_0) = E_0(\tau_1) + E_0[E_{X_{\tau_1}}(\sigma_0)] = \frac{1}{q_0} + \sum_{i=1}^{\infty} \frac{q_{0i}}{q_0} E_i(\sigma_0) = \frac{1}{\gamma} + \sum_{i=1}^{\infty} a_i E_i(\sigma_0).$$

Using (4.6) we have

$$E_0(\sigma_0) \leq \frac{1}{\gamma} + \frac{1}{\gamma \Gamma(\theta)} \left[\int_0^1 \frac{A(y)}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \right] \cdot \exp \left[\frac{1}{\Gamma(\theta)} \int_0^1 \frac{A(y)}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \right].$$

By (1.6), the right-hand side is finite. Thus the process is ergodic.

(3) By the assumption, there exists $C > 0$ such that $r_i \geq \frac{C}{b_0 - \Gamma} i$ for large enough i . Therefore

$$\begin{aligned}
\sum_{j=0}^{\infty} j q_{ij} &= \sum_{k=1}^{\infty} (i+k) r_i b_{k+1} + \sum_{k=1}^{\infty} (i+k) \gamma a_k + (i-1) r_i b_0 - (\gamma + r_i) i \\
&\leq m - r_i (b_0 - \Gamma) \leq m - C i.
\end{aligned}$$

Applying Corollary 4.49 in Chen (2004, p.150), we know the process is exponentially ergodic. \square

Proof of Theorem 1.8. (1) Using Lemma 5.1, we see the birth-death process (\hat{X}_t) is strongly ergodic. Let $u_i := E_i(T_0)$ for $i \geq 0$. Applying Theorem 4.44 and Lemma 4.48 in Chen (2004, p.145 and p.147), we find that (u_i) is a bounded non-negative solution to equation (5.7). By (5.9) and (5.10), (u_i) is also a non-negative bounded solution to the following equation

$$\sum_{j=1}^{\infty} q_{0j}u_j < \infty, \quad \sum_{j=0}^{\infty} q_{ij}u_j \leq -1, \quad i \geq 1.$$

By Theorem 4.45 in Chen (2004, p.145), we know the process is strongly ergodic.

(2) Suppose that (1.7) holds. Then

$$\int_0^1 \frac{A(y)}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \leq \gamma \int_0^1 \frac{1}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy < \infty.$$

Letting $i \rightarrow \infty$ in (4.6), we get

$$\sup_i E_i(\sigma_0) \leq \frac{1}{\Gamma(\theta)} \left[\int_0^1 \frac{1}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \right] \cdot \exp \left[\frac{1}{\Gamma(\theta)} \int_0^1 \frac{A(y)}{\alpha B(y)} \left(\ln \frac{1}{y} \right)^{\theta-1} dy \right] < \infty.$$

Then by Theorem 4.44 in Chen (2004, p.145) the process is strongly ergodic.

Conversely, suppose that X_t is strongly ergodic. By Theorem 4.44 in Chen (2004, p.145) and (4.7), we know (1.7) holds.

(3) By the strong Markov property, for $i \geq 1$ we have $E_i\sigma_0 = \sum_{k=1}^i E_k\sigma_{k-1}$. Notice that

$$E_k\sigma_{k-1} \geq E_k[\text{time spent at } k \text{ until the next jump}] = \frac{1}{r_k + \gamma}.$$

Thus $E_i\sigma_0 \geq \sum_{k=1}^i (r_k + \gamma)^{-1}$. By the assumption $\sum_{i=1}^{\infty} r_i^{-1} = \infty$, we have $\sup_i E_i\sigma_0 = \infty$. Applying Theorem 4.44 in Chen (2004, p.145), we know the process is not strongly ergodic. \square

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